

# The b-Chromatic Number of Regular Graphs via The Edge Connectivity

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## Abstract

The b-chromatic number of a graph  $G$ , denoted by  $\varphi(G)$ , is the largest integer  $k$  that  $G$  admits a proper coloring by  $k$  colors, such that each color class has a vertex that is adjacent to at least one vertex in each of the other color classes. El Sahili and Kouider [About b-colorings of regular graphs, Res. Rep. 1432, LRI, Univ. Orsay, France, 2006] asked whether it is true that every  $d$ -regular graph  $G$  of girth at least 5 satisfies  $\varphi(G) = d + 1$ . Blidia, Maffray, and Zemir [On b-colorings in regular graphs, Discrete Appl. Math. 157 (2009), 1787-1793] showed that the Petersen graph provides a negative answer to this question, and then conjectured that the Petersen graph is the only exception. In this paper, we investigate a strengthened form of the question.

The edge connectivity of a graph  $G$ , denoted by  $\lambda(G)$ , is the minimum cardinality of a subset  $U$  of  $E(G)$  such that  $G \setminus U$  is either disconnected or a graph with only one vertex. A  $d$ -regular graph  $G$  is called super-edge-connected if every minimum edge-cut is the set of all edges incident with a vertex in  $G$ , i.e.,  $\lambda(G) = d$  and every minimum edge-cut of  $G$  isolates a vertex. We show that if  $G$  is a  $d$ -regular graph that contains no 4-cycle, then  $\varphi(G) = d + 1$  whenever  $G$  is not super-edge-connected.

**Keywords:** b-chromatic number, edge connectivity, super-edge-connected.

**Subject classification:** 05C15

## 1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless, and without multiple edges). Let  $G = (V, E)$  be a graph. A (*proper vertex*) *coloring* of  $G$ , is a function  $f_G : V(G) \rightarrow C$  such that for each  $\{u, v\}$  in  $E(G)$ ,  $f_G(u) \neq f_G(v)$ . Each  $c$  in  $C$  is called a color. Also, for each  $c$  in  $C$ ,  $f_G^{-1}(c)$  is called a color class of  $f_G$ . We say  $v$  is colored by  $c$  if  $f_G(v) = c$ . We mean by  $\chi(G)$ , the minimum cardinality of a set  $C$  that a coloring  $f_G : V(G) \rightarrow C$  exists. A *b-coloring* of the graph  $G$  is a coloring  $f_G : V(G) \rightarrow C$  such that for each  $c$  in  $C$ , there exists some vertex  $v$  in  $V(G)$  such that  $f_G(v) = c$  and  $f_G(N_G(v)) = C \setminus \{c\}$ , where  $N_G(v) := \{w \mid \{v, w\} \in E(G)\}$ . In other words, a coloring of  $G$  is called a b-coloring, if each color class contains a vertex that is adjacent to at least one vertex in each of the other color classes. Obviously, each coloring of  $G$  with  $\chi(G)$  colors, is a b-coloring. Also, if  $f_G : V(G) \rightarrow C$  is a b-coloring, then  $|C| \leq \Delta(G) + 1$ , where  $\Delta(G)$  denotes the maximum degree of  $G$ . The *b-chromatic number* of  $G$ , denoted by  $\varphi(G)$ , is the maximum cardinality of a set  $C$  such that a b-coloring  $f_G : V(G) \rightarrow C$  exists. The concept of b-coloring of graphs introduced by Irving and Manlove in 1999 in [4], and has received attention.

Given a graph  $G$  and a coloring  $f_G : V(G) \rightarrow C$ , a vertex  $v$  in  $V(G)$  is called a *color-dominating* vertex (with respect to  $f_G$ ) if  $f_G(N_G(v)) = C \setminus \{f_G(v)\}$ . In other words,  $v$  is a color-dominating vertex, if it is adjacent to at least one vertex in each of the other color classes. We say that the color  $c$  *realizes*, if there exists a color-dominating vertex which is colored by  $c$ .

As mentioned, for each graph  $G$  with maximum degree  $\Delta(G)$ ,  $\varphi(G) \leq \Delta(G) + 1$ . Therefore, for each  $d$ -regular graph  $G$ ,  $\varphi(G) \leq d + 1$ . Since  $d + 1$  is the maximum possible b-chromatic number for  $d$ -regular graphs, determining necessary or sufficient conditions to achieve this bound is of interest. Kratochvil, Tuza, and Voigt [7] proved that every  $d$ -regular graph  $G$  with at least  $d^4$  vertices satisfies  $\varphi(G) = d + 1$ . In [2], Cabello and Jakovac lowered  $d^4$  to  $2d^3 - d^2 + d$ . These amazing bounds confirm that for each natural number  $d$ , there are only a finite number of  $d$ -regular graphs (up to isomorphism) that their b-chromatic numbers are not  $d + 1$ . El Sahili and Kouider [3] asked whether it is true that every  $d$ -regular graph  $G$  of girth at least 5 satisfies  $\varphi(G) = d + 1$ . In this regard, Blidia, Maffray, and Zemir [1] showed that the Petersen graph provides a negative answer to this question. They proved that the b-chromatic number of the Petersen graph is 3, and then conjectured that the Petersen graph is the only exception. They also proved this conjecture for  $d \leq 6$ . In [6], Kouider proved that the b-chromatic number of any  $d$ -regular graph of girth at least 6 is  $d + 1$ . El Sahili and Kouider [3] showed that the b-chromatic number of any  $d$ -regular graph of girth 5 that contains no 6-cycle is  $d + 1$ . In [2], Cabello and Jakovac proved a celebrated theorem for the b-chromatic number of regular graphs of girth 5, which guarantees that the b-chromatic number of each  $d$ -regular graph with girth 5, is bounded below by a linear function of  $d$ . They proved that a  $d$ -regular graph with girth at least 5, has b-chromatic number at least  $\lfloor \frac{d+1}{2} \rfloor$ . Also, they proved that for except small values of  $d$ , every connected  $d$ -regular graph that contains no 4-cycle and its diameter is at least  $d$ , has b-chromatic number  $d + 1$ . It is shown in [8] that if  $G$  is a  $d$ -regular graph that contains no 4-cycle, then  $\varphi(G) \geq \lfloor \frac{d+3}{2} \rfloor$ . This lower bound, is sharp for the Petersen graph. Besides, If  $G$  has a triangle, then  $\varphi(G) \geq \lfloor \frac{d+4}{2} \rfloor$ . Also, if  $G$  is a  $d$ -regular graph that contains no 4-cycle and  $\text{diam}(G) \geq 6$ , then  $\varphi(G) = d + 1$ .

The *vertex connectivity* of a graph  $G$ , denoted by  $\kappa(G)$ , is the minimum cardinality of a subset  $U$  of  $V(G)$  such that  $G \setminus U$  is either disconnected or a graph with only one vertex. Also, the *edge connectivity* of a graph  $G$ , denoted by  $\lambda(G)$ , is the minimum cardinality of a subset  $U$  of  $E(G)$  such that  $G \setminus U$  is either disconnected or a graph with only one vertex. It is well-known that for each graph  $G$ ,  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ , where  $\delta(G)$  denotes the minimum degree of  $G$ . By an *edge-cut* of  $G$ , we mean a subset of  $E(G)$  such that deleting all of its elements from  $G$ , yields a disconnected graph. Therefore, for each graph  $G$  with at least two vertices,  $\lambda(G)$  is the minimum cardinality of all edge-cuts of  $G$ . For every edge-cut  $T$  of  $G$ ,  $\text{sat}(T)$  stands for the set of all vertices of  $G$  that are saturated by some edges in  $T$ , i.e.,  $\text{sat}(T) := \{v \mid v \in V(G), \text{ there exists some } w \text{ in } V(G) \text{ for which } \{v, w\} \in T\}$ . We mean by a minimum edge-cut of  $G$ , an edge-cut of  $G$  with cardinality  $\lambda(G)$ . A  $d$ -regular graph  $G$ , is called *super-edge-connected*, if every minimum edge-cut of  $G$ , is the set of all edges incident with a vertex in  $G$ , i.e.,  $\lambda(G) = d$  and deleting each minimum edge-cut of  $G$  from  $G$ , yields a graph which has an isolated vertex.

An edge-cut of a graph  $G$  is called *trivial* whenever it is equal to the set of all edges incident with a vertex of  $G$ . With this terminology, a  $d$ -regular graph  $G$  is super-edge-connected if and only if every minimum edge-cut of  $G$  is trivial.

It has been proved in [8] that for any  $d$ -regular graph  $G$  that contains no 4-cycle, if  $\kappa(G) \leq \frac{d+1}{2}$ , then  $\varphi(G) = d + 1$ . This upper bound is sharp in the sense that the vertex connectivity of the Petersen graph is  $\frac{d+1}{2} + 1$ ; nevertheless, its b-chromatic number is not  $d + 1$ . Also, if  $\kappa(G) < \frac{3d-3}{4}$ , then  $\min\{2(d - \kappa(G) + 1), d + 1\} \leq \varphi(G) \leq d + 1$ . Furthermore, if there exists a subset  $U$  of  $V(G)$  such that  $|U| = \kappa(G)$  and  $G \setminus U$  has at least four connected components, then  $\varphi(G) = d + 1$ . Moreover, if  $\kappa(G) < \frac{2d-1}{3}$  and there exists a subset  $U$  of  $V(G)$  such that  $|U| = \kappa(G)$  and  $G \setminus U$  has at least three connected components, then  $\varphi(G) = d + 1$ . If the girth of  $G$  is 5,  $\frac{3d-3}{4}$  and  $\frac{2d-1}{3}$  can be replaced by  $\frac{3d}{4}$  and  $\frac{2d+1}{3}$ , respectively.

In this paper, we investigate the b-chromatic number of  $d$ -regular graphs with no 4-cycles. We show that if  $G$  is a  $d$ -regular graph that contains no 4-cycle, then  $\varphi(G) = d + 1$  whenever  $G$  is not super-edge-connected. Throughout the paper, for each nonnegative integer  $n$ , the symbol  $[n]$  stands for the set  $\{i \mid i \in \mathbb{N}, 1 \leq i \leq n\}$ .

## 2 The Main Result

This section concerns a relation between the b-chromatic number and the edge-connectivity in  $d$ -regular graphs that do not contain 4-cycles. We show that every  $d$ -regular graph that does not contain  $C_4$  as a subgraph, achieves the maximum b-chromatic number  $d + 1$ , unless it is super-edge-connected. In this regard, first we mention Lemma 1 and Lemma 2; the former is related to the super-edge-connected graphs without 4-cycle, and the latter presents a sufficient condition for bipartite graphs to have a perfect matching.

**Lemma 1.** *Let  $d \geq 4$  and  $G$  be a  $d$ -regular graph that contains no 4-cycle, and  $T$  be a minimum edge-cut of  $G$  which is not trivial. Suppose that  $G_1, \dots, G_l$  are connected components of  $G \setminus T$ . Then for each  $i$  in  $[l]$ , there exists some  $a_i$  in  $V(G_i) \setminus \text{sat}(T)$  such that  $|N_G(a_i) \cap \text{sat}(T)| \leq 2$ .*

**Proof.** Let us regard an arbitrary  $i$  in  $[l]$  as fixed. Set  $A_i := V(G_i) \cap \text{sat}(T)$  and  $s := |V(G_i)|$ . Since  $T$  is not trivial,  $s \geq 2$ . Obviously,  $s \neq 2$ ; otherwise, the number of edges between  $V(G_i)$  and  $V(G) \setminus V(G_i)$  in the graph  $G$  is at least  $2(d - 1)$ . So  $|T| \geq 2(d - 1) \geq d + 2$ , a contradiction. Hence,  $s \geq 3$ . It is well-known that the number of edges of a graph with  $n$  vertices which contains no 4-cycle is at most  $\frac{n}{4}(1 + \sqrt{4n - 3})$ . Since the graph  $G_i$  does not contain any 4-cycles,  $sd = \sum_{x \in V(G_i)} \deg_G(x) \leq \frac{s}{2}(1 + \sqrt{4s - 3}) + |T| \leq \frac{s}{2}(1 + \sqrt{4s - 3}) + d$ . Hence,

$$\begin{aligned} (s - 1)d &\leq \frac{s}{2}(1 + \sqrt{4s - 3}) \implies 2(s - 1)d - s \leq s\sqrt{4s - 3} \\ &\implies (2d - 1)(s - 1) - 1 \leq s\sqrt{4s - 3} \\ &\implies (2d - 2)(s - 1) < s\sqrt{4s - 3} \quad (\text{since } 2 < s) \\ &\implies d - 1 < \frac{s}{2(s - 1)}\sqrt{4s - 3}. \end{aligned}$$

One can easily observe that the derivative of the function  $f(x) = \frac{x}{2(x-1)}\sqrt{4x-3} - (d-1)$  is positive for  $x \in [3, +\infty)$ . So  $f$  is strictly increasing in the interval  $[3, +\infty)$ .

Thus, proving  $f(d+3) < 0$ , implies  $s \geq d+4$ . Since the derivative of the function  $g(y) = \frac{y+3}{2(y+2)}\sqrt{4y+9} - (y-1)$  is negative for  $y \in (0, +\infty)$  and  $g(4) = \frac{-1}{3} < 0$ , we obtain that for each natural number  $d$  which  $d \geq 4$ ,  $g(d) < 0$ . Hence,  $f(d+3) = g(d) < 0$ ; and therefore,  $s \geq d+4$ . Since  $|A_i| \leq |T| \leq d$ ,  $|V(G_i) \setminus A_i| \geq 4$ . Now, consider four elements  $x_1, x_2, x_3$ , and  $x_4$ , in  $V(G_i) \setminus A_i$ . Since  $G$  does not have any 4-cycles, there exists some  $j$  in  $\{1, 2, 3, 4\}$  such that  $|N_G(x_j) \cap A_i| < d$ . So  $|N_G(x_j) \setminus A_i| > 0$ . Set  $N_G(x_j) \setminus A_i = \{y_k | 1 \leq k \leq |N_G(x_j) \setminus A_i|\}$ . Obviously, there exists some  $k$  in  $[|N_G(x_j) \setminus A_i|]$  such that  $|N_G(y_k) \cap (A_i \setminus N_G(x_j))| \leq 1$ ; otherwise,  $d \geq |A_i| \geq |N_G(x_j) \cap A_i| + 2|N_G(x_j) \setminus A_i| = d + |N_G(x_j) \setminus A_i| > d$ , which is impossible. We conclude that there exists an element  $y_k$  in  $N_G(x_j) \setminus A_i$  for which  $|N_G(y_k) \cap (A_i \setminus N_G(x_j))| \leq 1$ . Since  $y_k$  has at most one neighbor in  $N_G(x_j) \cap A_i$ ; hence,  $y_k$  has at most two neighbors in  $A_i$ . Therefore,  $a_i := y_k$  is a desired vertex.  $\blacksquare$

**Lemma 2.** [2] *Let  $H$  be a bipartite graph with parts  $U$  and  $V$  such that  $|U| = |V|$ . Let  $u^* \in U$  and  $v^* \in V$ . If for each vertex  $x$  in  $V(H) \setminus \{u^*, v^*\}$ ,  $\deg_H(x) \geq \frac{|V|}{2}$ ,  $\deg_H(u^*) > 0$ , and  $\deg_H(v^*) > 0$ , then  $H$  has a perfect matching.*

We are now in a position to prove the main result of the paper.

**Theorem 1.** *Let  $G$  be a  $d$ -regular graph that contains no 4-cycle. If  $G$  is not super-edge-connected, then  $\varphi(G) = d+1$ .*

**Proof.** There is nothing to prove when  $d \in \{0, 1, 2\}$ . Also, Jakovac and Klavžar, in [5], showed that the only cubic graph that contains no 4-cycle and its b-chromatic number is not equal to 4, is the Petersen graph. Since the Petersen graph is super-edge-connected, the proof is completed for  $d = 3$ . So we suppose that  $d \geq 4$ . Since  $G$  is not super-edge-connected, there exists a minimum edge-cut  $T$  of  $G$  which is not trivial. Suppose that  $G_1, \dots, G_l$  are connected components of  $G \setminus T$ . For each  $i$  in  $[l]$ , define  $A_i := \{x | x \in V(G_i), \exists y \in V(G) \setminus V(G_i) \text{ such that } \{x, y\} \in T\}$ . According to the Lemma 1, for each  $i$  in  $[l]$ , there exists some  $a_i$  in  $V(G_i) \setminus A_i$  for which  $|N_G(a_i) \cap A_i| \leq 2$ .

Let  $x_1, \dots, x_{d-|N_G(a_1) \cap A_1|}$  be an arbitrary ordering of all elements of  $N_G(a_1) \setminus A_1$ . Color the vertex  $a_1$  by color 1, and for each  $i$  in  $[|N_G(a_1) \setminus A_1|]$ , assign the color  $i+1$  to the vertex  $x_i$ . Also, color all vertices that are in the set  $N_G(a_1) \cap A_1$  by all colors that are in the set  $[d+1] \setminus [|N_G(a_1) \setminus A_1| + 1]$  injectively. Then, color all vertices that are in the set  $N_G(a_2) \cap A_2$  by some colors that are in the set  $\{1, \lfloor \frac{d+2}{2} \rfloor\}$  injectively.

The vertex  $a_1$  is a color-dominating vertex with color 1. Now, our task is to color all the vertices in  $(\bigcup_{i=1}^{\lfloor \frac{d}{2} \rfloor} N_G(x_i)) \setminus (\{a_1\} \cup N_G(a_1))$  by colors in  $[d+1]$  such that for each  $i$  in  $[\frac{d}{2}]$ , all colors that are in the set  $[d+1] \setminus \{i+1\}$ , appear on  $N_G(x_i)$ .

For each  $i$  in  $[\lfloor \frac{d}{2} \rfloor]$ , set  $V_i, S_i$ , and  $C_i$ , as follows:

- $V_i := N_G(x_i) \setminus (\{a_1\} \cup N_G(a_1))$ ;
- $S_i := \{a_1\} \cup N_G(a_1) \cup (\bigcup_{j=1}^i V_j) \cup (N_G(a_2) \cap A_2)$ ;
- $C_i := ([d+1] \setminus \{1\}) \setminus (\text{the set of colors that were appeared on } \{x_i\} \cup (N_G(x_i) \cap N_G(a_1)))$ .

Since  $G$  contains no 4-cycle, for any two distinct natural numbers  $i$  and  $j$  in  $[\lfloor \frac{d}{2} \rfloor]$ ,  $V_i \cap V_j = \emptyset$ . Also, since  $G$  contains no 4-cycle, the maximum degree of the induced subgraph of  $G$  on  $N_G(a_1)$  is at most one. So  $|V_i| = |C_i| = d-1$  or  $|V_i| = |C_i| = d-2$ . Moreover,  $|V_i| = |C_i| = d-2$  if and only if  $|N_G(x_i) \cap N_G(a_1)| = 1$ . Now, we follow  $\lfloor \frac{d}{2} \rfloor$  steps inductively. For each  $i$  in  $[\lfloor \frac{d}{2} \rfloor]$ , at  $i$ -th step, we only color all vertices that are in  $V_i$  by all colors that are in  $C_i$  injectively. Suppose by induction that  $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$  and for each  $k$  in  $[i-1]$ , at  $k$ -th step, we have only colored all vertices that are in  $V_k$  by all colors that are in  $C_k$  injectively in such a way that the resulting partial coloring on  $S_k$  is a proper coloring. Now, at  $i$ -th step, we want to color only all vertices that are in  $V_i$  by all colors that are in  $C_i$  injectively such that the resulting partial coloring on  $S_i$  be a proper partial coloring. Consider a bipartite graph  $H_i$  with one part  $V_i$  and the other part  $C_i$ , which a vertex  $v$  in  $V_i$  is adjacent to a color  $c$  in  $C_i$  in the graph  $H_i$  if and only if (in the graph  $G$ )  $v$  does not have any neighbors in  $S_i$  already colored by  $c$ . Such a coloring of all vertices that are in  $V_i$  by all colors that are in  $C_i$  (as mentioned) exists if and only if  $H_i$  has a perfect matching.

Let  $v$  be an arbitrary element of  $V_i$ . The set of neighbors of  $v$  in the graph  $G$  that were already colored, is a subset of  $\{x_i\} \cup (\bigcup_{j=1}^{i-1} V_j) \cup (N_G(a_2) \cap A_2)$ . Since  $G$  contains no 4-cycle, for each  $j$  in  $[i-1]$ ,  $v$  has at most one neighbor in  $V_j$ . Also,  $v$  has at most one neighbor in  $N_G(a_2) \cap A_2$ . However, the color of the vertex  $x_i$  does not belong to  $C_i$ . Therefore,  $\deg_{H_i}(v) \geq |C_i| - i$ . Also, since for each  $j$  in  $[i-1]$ ,  $v_j$  sees all colors of  $[d+1]$  on its closed neighborhood, each color of  $[d+1]$  appears at most once on  $V_j$ . Besides, each color of  $[d+1]$  appears at most once on  $N_G(a_2) \cap A_2$ . Therefore, for each  $c$  in  $C_i$ ,  $\deg_{H_i}(c) \geq |V_i| - i$ . Hence, for each  $v$  in  $V(H_i)$ ,  $\deg_{H_i}(v) \geq |V_i| - i$ . Since  $|V_i| \geq d-2$ , if  $1 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$ , then  $\deg_{H_i}(v) \geq |V_i| - i \geq \lfloor \frac{|V_i|}{2} \rfloor$ . In the case  $i = \lfloor \frac{d}{2} \rfloor$ , since the color of each vertex in  $N_G(a_2) \cap A_2$  belongs to the set  $\{1, \lfloor \frac{d+2}{2} \rfloor\}$  and also  $C_{\lfloor \frac{d}{2} \rfloor} \cap \{1, \lfloor \frac{d+2}{2} \rfloor\} = \emptyset$ , for each  $v$  in  $V(H_i)$ ,  $\deg_{H_i}(v) \geq |V_i| - (i-1) \geq \lfloor \frac{|V_i|}{2} \rfloor$ .

So Lemma 2 implies that  $H_i$  has a perfect matching and we are done. We conclude that there exists a partial coloring on  $S_{\lfloor \frac{d}{2} \rfloor}$  by all colors of the set  $[d+1]$ , such that  $a_1, x_1, \dots, x_{\lfloor \frac{d}{2} \rfloor}$  are color-dominating vertices whose colors are  $1, 2, \dots, \lfloor \frac{d+2}{2} \rfloor$ , respectively.

The next task is to color some uncolored vertices from  $V(G_2)$  in such a way that all colors in  $[d+1] \setminus [\lfloor \frac{d+2}{2} \rfloor]$  realize. The procedure is to find  $\lfloor \frac{d-1}{2} \rfloor$  suitable vertices  $z_1, \dots, z_{\lfloor \frac{d-1}{2} \rfloor}$  in  $N_G(a_2) \setminus A_2$  in order to make them along  $a_2$  color-dominating.

Let  $N_G(a_2) \setminus A_2 = \{y_i | 1 \leq i \leq |N_G(a_2) \setminus A_2|\}$ . For each  $i$  in  $[|N_G(a_2) \setminus A_2|]$ , set  $W_i$  and  $e_{y_i}$  as follows:

- $W_i := N_G(y_i) \setminus (\{a_2\} \cup N_G(a_2))$ ;
- $e_{y_i} := |\{\{s, t\} | \{s, t\} \in E(G), s \in W_i, t \in A_1\}|$ .

Since  $G$  does not contain any 4-cycles, for any two different natural numbers  $i$  and  $j$  in  $[|N_G(a_2) \setminus A_2|]$ ,  $W_i \cap W_j = \emptyset$ . Without loss of generality, we can assume that the sequence  $\{e_{y_i}\}_{i=1}^{|N_G(a_2) \setminus A_2|}$  is decreasing, i.e.,  $e_{y_1} \geq \dots \geq e_{y_{|N_G(a_2) \setminus A_2|}}$ . We show that for each natural number  $i$ ,  $|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + 1 \leq i \leq |N_G(a_2) \setminus A_2|$ ,  $e_{y_i} \leq 1$ .

Suppose, on the contrary, that for some natural number  $i$ ,  $|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + 1 \leq i \leq |N_G(a_2) \setminus A_2|$ ,  $e_{y_i} > 1$ . Therefore, for each  $j$  in  $[|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + 1]$ ,  $e_{y_j} \geq 2$ . Since each vertex in  $N_G(a_2) \cap A_2$  is incident with an edge of  $T$ , so

$$\begin{aligned} d \geq |T| &\geq |N_G(a_2) \cap A_2| + 2(|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + 1) = \\ &|N_G(a_2) \cap A_2| + 2(d - |N_G(a_2) \cap A_2| - \lfloor \frac{d-3}{2} \rfloor) \geq \\ &d + 3 - |N_G(a_2) \cap A_2| \geq d + 3 - 2 = d + 1, \end{aligned}$$

which is impossible. Accordingly, for each natural number  $i$ ,  $|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + 1 \leq i \leq |N_G(a_2) \setminus A_2|$ ,  $e_{y_i} \leq 1$ . For each  $i$  in  $[\lfloor \frac{d-1}{2} \rfloor]$ , put  $z_i := y_{|N_G(a_2) \setminus A_2| - \lfloor \frac{d-1}{2} \rfloor + i}$ ; therefore,  $e_{z_i} \leq 1$ .

Now, color the vertex  $a_2$  by color  $d + 1$  and for each  $i$ ,  $1 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$ , assign the color  $\lfloor \frac{d+2}{2} \rfloor + i$  to the vertex  $z_i$ . Also, color all vertices that are in the set  $N_G(a_2) \setminus (A_2 \cup \{z_i | 1 \leq i \leq \lfloor \frac{d-1}{2} \rfloor\})$  by some colors of  $[d + 1]$  injectively in such a way that all colors of the set  $[d + 1]$  appear on the closed neighborhood of  $a_2$ .

For each  $i$  in  $[\lfloor \frac{d-1}{2} \rfloor]$ , define  $V'_i$ ,  $S'_i$ , and  $C'_i$ , as follows:

- $V'_i := N_G(z_i) \setminus (\{a_2\} \cup N_G(a_2))$ ;
- $S'_i := \{a_2\} \cup N_G(a_2) \cup (\bigcup_{j=1}^i V'_j) \cup S_{\lfloor \frac{d}{2} \rfloor}$ ;
- $C'_i := [d] \setminus (\text{the set of colors that were appeared on } \{z_i\} \cup (N_G(z_i) \cap N_G(a_2)))$ .

The maximum degree of the induced subgraph of  $G$  on  $N_G(a_2)$  is at most one. So  $|V'_i| = |C'_i| = d - 1$  or  $|V'_i| = |C'_i| = d - 2$ . Furthermore,  $|V'_i| = |C'_i| = d - 2$  if and only if  $|N_G(z_i) \cap N_G(a_2)| = 1$ . Now, we follow  $\lfloor \frac{d-1}{2} \rfloor$  steps inductively. For each  $i$  in  $[\lfloor \frac{d-1}{2} \rfloor]$ , at  $i$ -th step, we only color all vertices that are in  $V'_i$  by all colors that are in  $C'_i$  injectively. Suppose by induction that  $1 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$  and for each  $k$  in  $[i - 1]$ , at  $k$ -th step, we have only colored all vertices that are in  $V'_k$  by all colors in  $C'_k$  injectively in such a way that the resulting partial coloring on  $S'_k$  is a proper coloring. Now, at  $i$ -th step, we aim to color only all vertices that are in  $V'_i$  by all colors that are in  $C'_i$  injectively such that the resulting partial coloring on  $S'_i$  be a proper partial coloring. Consider a bipartite graph  $H'_i$  with one part  $V'_i$  and the other part  $C'_i$ , that a vertex  $v$  in  $V'_i$  is adjacent to a color  $c$  in  $C'_i$  in the graph  $H'_i$ , if and only if (in the graph  $G$ )  $v$  does not have any neighbors in  $S'_i$  already colored by  $c$ . Such a coloring of all vertices that are in  $V'_i$  by all colors that are in  $C'_i$  exists if and only if  $H'_i$  has a perfect matching. So our goal is to prove the existence of a perfect matching in  $H'_i$ . In this regard, we again apply Lemma 2.

Since  $e_{z_i} \leq 1$ , there exists at most one edge between  $V'_i$  and  $A_1$  in the graph  $G$ . Thus, there exists at most one element in  $V'_i$  that has a neighbors in  $V(G_1)$  (in the graph  $G$ ). If such a vertex exists, call it  $v^*$ . Therefore, in the graph  $H'_i$ , the degree of each vertex in  $V'_i \setminus \{v^*\}$  is at least  $|C'_i| - (i - 1)$ . Similarly, there exists at most one color in  $C'_i$  for which there is an element in  $V'_i$  that has a neighbor in  $V(G_1)$  with this color. If such a color exists, call it  $c^*$ . Hence, in the graph  $H'_i$ , the degree of each vertex in  $C'_i \setminus \{c^*\}$  is at least  $|V'_i| - (i - 1)$ . We conclude that for each vertex  $x$  in  $V_{H'_i} \setminus \{v^*, c^*\}$ ,  $\deg_{H'_i}(x) \geq |V'_i| - (i - 1)$ ; and since  $i \leq \lfloor \frac{d-1}{2} \rfloor$  and  $|V'_i| \geq d - 2$ ,

$\deg_{H'_i}(x) \geq \frac{|V'_i|}{2}$ . Also, the degree of each vertex of  $H'_i$  is at least  $|V'_i| - i$  which is positive.

We conclude that  $|V'_i| = |C'_i|$  and the degree of each vertex in  $V_{H'_i} \setminus \{v^*, c^*\}$  is at least  $\frac{|V'_i|}{2}$ . Also, the degree of each vertex of  $H'_i$  is positive. Accordingly, the Lemma 2 implies that  $H'_i$  has a perfect matching. Therefore, there exists a partial coloring on  $S'_{\lfloor \frac{d-1}{2} \rfloor}$  by all colors of the set  $[d+1]$  such that all colors realize. This partial coloring can be extended to a coloring of the graph  $G$  greedily. So  $\varphi(G) = d+1$ . ■

**Acknowledgements:** I am grateful to Professor Hossein Hajiabolhassan for his many invaluable comments. Also, I would like to express my gratitude to Professor Rashid Zaare-Nahandi for his support during my M.Sc. and Ph.D. education.

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